1 KL Expansion for Compound Poisson Process (Parikshit)

$$
\begin{aligned}
R(s,t) &= \lambda\sigma^2 \min(s,t) + st(\lambda\mu)^2, &(1)\\
\Lambda_n\phi_n(s) &= (\lambda\mu)^2 s\int_0^T t\phi_n(t)\,dt + \lambda\sigma^2\int_0^s t\phi_n(t)\,dt + \lambda\sigma^2 s\int_s^T \phi_n(t)\,dt, &(2)\\
\Lambda_n\phi_n'(s) &= (\lambda\mu)^2\int_0^T t\phi_n(t)\,dt + \lambda\sigma^2\left[s\phi_n(s) + \int_0^s \frac{\partial(t\phi_n(t))}{\partial s}\,dt\right]\\
&\quad +\lambda\sigma^2\int_s^T \phi_n(t)\,dt + \lambda\sigma^2 s\left[-\phi_n(s) + \int_s^T \frac{\partial(\phi_n(t))}{\partial s}\,dt\right], \quad \text{differential w.r.t. } s &(3)\\
\Lambda_n\phi_n'(s) &= (\lambda\mu)^2\int_0^T t\phi_n(t)\,dt + \lambda\sigma^2\int_s^T \phi_n(t)\,dt, &(4)\\
\Lambda_n\phi_n''(s) &= -\lambda\sigma^2\phi_n(s), \qquad \text{Let } \frac{\Lambda_n}{\lambda\sigma^2} =: \beta_n^2, &(5)\\
\Rightarrow \phi_n(t) &= A\sin\left(\frac{t}{\beta_n}\right) + B\cos\left(\frac{t}{\beta_n}\right) &(6)\\
\Lambda_n\phi_n(0) &= 0, \qquad B = 0, &(7)\\
\int_0^T \phi_n^2(t)\,dt &= 1 \Rightarrow A\int_0^T \sin^2\left(\frac{t}{\beta_n}\right)\,dt = 1, &(8)\\
\Rightarrow A &= \frac{2}{\sqrt{\left[2T - \beta_n\sin\frac{2T}{\beta_n}\right]}}, &(9)\\
\Rightarrow \phi_n(t) &= \frac{2}{\sqrt{\left[2T - \beta_n\sin\frac{2T}{\beta_n}\right]}}\sin\left(\frac{t}{\beta_n}\right), \qquad \beta_n = \sqrt{\frac{\Lambda_n}{\lambda\sigma^2}}, &(10)\\
&\text{substituting } s = T \text{ in the first deriv. and then the value of the Poisson eigfn.} &(11)\\
\frac{\Lambda_n}{\beta_n}\cos\frac{T}{\beta_n} &= (\lambda\mu)^2\beta_n^2\sin\frac{T}{\beta_n} - (\lambda\mu)^2 T\beta_n\cos\frac{T}{\beta_n}, \qquad \text{transcendental eqn in } \Lambda_n &(12)\\
&\text{substituting } s = T \text{ in the first eqn and then the value of the Poisson eigfn.} &(13)\\
\Lambda_n\sin\frac{T}{\beta_n} &= \left[(\lambda\mu)^2 T + \lambda\sigma^2\right]\left[\beta_n^2\sin\frac{T}{\beta_n} - T\beta_n\cos\frac{T}{\beta_n}\right], \qquad \text{transcendental eqn in } \Lambda_n &(14)
\end{aligned}
$$

(Abhishek) Since $\beta_n = \sqrt{\frac{\Lambda_n}{\lambda\sigma^2}}$, solving for eigenvalues Λ_n boils down to solve for β_n as a function of the parameters $\lambda, \sigma, \mu, T > 0$, and $n \in \mathbb{N}$. From (12) and (14), we have

$$
\begin{aligned}
\frac{1}{(\lambda\mu)^2}\frac{\Lambda_n}{\beta_n}\cos\frac{T}{\beta_n} &= \frac{1}{(\lambda\mu)^2 T + \lambda\sigma^2}\Lambda_n\sin\frac{T}{\beta_n}, &(15)\\
\Rightarrow \beta_n\tan\frac{T}{\beta_n} &= T + \frac{\sigma^2}{\lambda\mu^2}, \qquad \text{since } \Lambda_n \neq 0,\ \forall n\in\mathbb{N}, &(16)\\
\Rightarrow \tan x &= m\,x, \qquad \text{where } x \triangleq \frac{T}{\beta_n}, \text{ and } m \triangleq 1 + \frac{1}{\lambda T}\left(\frac{\sigma}{\mu}\right)^2. &(17)
\end{aligned}
$$

Thus, solving for $x > 0$ is same as finding *positive abscissa* for intersections of $\tan x$ and a straight line passing through the origin with slope $> 45°$ (since $m > 1$, from (17)). Such intersections happen in *either first or fourth quadrant*, depending on the value of m. Hence, x will be a function of $(2n-1)\frac{\pi}{2}$, up to translation. Consequently, the KL expansion of compound Poisson process $Y(\omega, t)$ is given by

$$
Y(\omega,t) = \sum_{n=1}^{\infty}\sqrt{\Lambda_n}\zeta_n(\omega)\phi_n(t), \tag{18}
$$

where Λ_n solves

$$
\tan\left(\sigma T\sqrt{\frac{\lambda}{\Lambda_n}}\right) = \left[1 + \frac{1}{\lambda T}\left(\frac{\sigma}{\mu}\right)^2\right]\left(\sigma T\sqrt{\frac{\lambda}{\Lambda_n}}\right), \qquad \lambda, \sigma, \mu, T > 0; n\in\mathbb{N}. \tag{19}
$$

Further, $\zeta_n(\omega)$ are i.i.d random variables from $\mathcal{N}(0,1)$, and the eigenfunctions $\phi_n(t) = \frac{2}{\sqrt{\left[2T - \beta_n \sin\frac{2T}{\beta_n}\right]}} \sin\left(\frac{t}{\beta_n}\right)$, $\beta_n \triangleq \sqrt{\frac{\Lambda_n}{\lambda\sigma^2}}$.

2 KL Expansion for Poisson White Noise

Taking the time derivative of the KL expansion of compound Poisson process in m.s. sense, we get the KL expansion for Poisson white noise $Z(\omega, t)$ as

$$Z(\omega, t) = \sum_{n=1}^{\infty} \sqrt{\Lambda_n}\zeta_n(\omega) \frac{\frac{2}{\beta_n}}{\sqrt{\left[2T - \beta_n \sin\frac{2T}{\beta_n}\right]}} \cos\left(\frac{t}{\beta_n}\right), \tag{20}$$

where the Λ_n, $\zeta_n(\omega)$ and β_n are as in the preceding section.

Remark 1 *Setting the parameters $\lambda = 1$, $\mu = 0$, in (1), we recover Wiener process as a special case of compound Poisson process. Substituting the same in (12), we indeed recover the well-known eigenvalues and eigenfunctions of the covariance kernel of Wiener process, and consequently, the KL expansion of Gaussian white noise.*

Remark 2 *In this document, we used the following definition of compound Poisson process $Y(\omega, t)$:*

$$Y(\omega, t) = \begin{cases} 0, & \text{if } N(t) = 0, \\ \sum_{j=1}^{N(t)} Y_j(\omega), & \text{if } N(t) > 0, \end{cases} \tag{21}$$

where $N(t)$ is a homogeneous Poisson counting process with intensity parameter $\lambda > 0$, and $Y_j(\omega)$ are i.i.d random variables drawn from $\mathcal{N}(\mu, \sigma^2)$. The choice of Gaussian distribution is a working convenience. Instead of Gaussian, one can take more general distribution for Y_j. In that case, $Y(\omega, t)$ is still called compound Poisson process as long as the chosen distribution for Y_j is independent to that of the counting process $\{N(t)\}_{t \geqslant 0}$.