Frequency Domain Model Validation in Wasserstein Metric

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Abstract— This paper connects the time-domain uncertainty propagation approach for model validation in Wasserstein distance $_2W_2$, introduced by the authors in [1], with the frequency domain model validation in the same. To the best of our knowledge, this is the first frequency domain interpretation of Monge-Kantorovich optimal transport. It is shown that the asymptotic $_2W_2$ can be written as functions of the $\mathcal{H}_2$ norms of the system gains, which have intuitive meaning. A geometric interpretation for this newly derived frequency-domain formula is given. The geometric interpretation helps us in comparing Wasserstein distance with classical frequency-domain validation metrics like ν-gap.

I. INTRODUCTION

A probabilistic formulation for the *time-domain* model validation problem was proposed recently [1] by the authors. For both deterministic and stochastic systems, the proposed method uses uncertainty propagation as a construct to perform model validation. The main idea relies on the following notion: two systems, starting from the same initial ensemble, are close if the distributions of trajectories supported over the respective output spaces, remain "close" at each time. The notion of distributional closeness, as opposed to closeness in trajectories, is desired to account various sources of uncertainties (e.g. initial condition, parametric, modeling uncertainty etc.). Moreover, it is desired that the validation framework should not make any prior assumption about the nature of the uncertainty (e.g. interval-valued structured uncertainty as in robust control based validation frameworks [2], [3], set-valued uncertainty as in barrier certificate framework [4], statistically parametric (moment-based) uncertainty as in polynomial chaos framework [5]). To quantify the distributional closeness between the measured and model-predicted outputs, we used L_2 Wasserstein distance of order 2, denoted as $_2W_2$ as a metric. It was shown in [1] that unlike pointwise notions of distances on the manifold of probability densities, the Wasserstein distance, originated from the *Monge-Kantorovich theory of optimal transport* [6], [7], is an integral notion of distance and is a measure of the minimum amount of work needed to convert one distributional shape to the other. It was shown that computation of $_2W_2$ is fully non-parametric and it is robust against unequal number of samples, and non-overlapping supports – situations commonly encountered in nonlinear model validation. To conclude a model is valid with high probability, its Wasserstein gap trajectory $_2W_2(t)$ must lie below user-specified tolerance levels, at all times. A probabilistically robust validation certificate was constructed

in [1] to guarantee provably correct inference through a finite-sample computation.

Although the time-domain formulation described above provides a unifying model validation framework in both discrete and continuous time, for both linear and nonlinear systems, with heterogeneous sources of uncertainties, two fundamental question remains: for the special case of linear model validation, is there a frequency-domain characterization of Wasserstein gap? If yes, then how does that compare with the existing notions of frequency domain model validation?

The purpose of this paper is to answer these two questions. We provide frequency domain characterizations for the asymptotic Wasserstein distance $\lim_{t\to\infty} {}_2W_2(t)$, henceforth simply denoted as W, in terms of the $\mathcal{H}_2$ norms of the system transfer gains. We derive exact frequency-domain formulae for W for single-input-single-output (SISO), multi-input-single-output (MISO), and multi-input-multi-output (MIMO) case. For SISO and MISO case, the formulae for W have a nice intuitive interpretation as the difference between average system gains. For MIMO case, though exact frequency-domain formula is complicated than SISO and MISO case, we derive simpler upper and lower bounds for the same.

To answer the second question, we resort to geometric arguments via stereographic projection. To compare ν-gap metric (which is normalized between 0 and 1) and W (which is un-normalized), we provide two different constructions. One is un-normalized comparison, which results comparison on the extended complex plane and the other is normalized comparison, which results comparison on the Riemann sphere. During the second comparison, we get an intrinsic normalization of W. Some properties about the sensitivity of W as a frequency-domain metric are discussed.

Notation: Unless otherwise specified, a transfer function with "hat" symbol $\hat{}$ denotes the model, else it denotes the true system. Superscripts $*$ and ^{H} denote complex conjugate and conjugate transpose respectively. The notation tr(.) stands for trace, det(.) stands for determinant, and $||M||_F$ denotes the Frobenius norm of matrix M. The symbol wno(.) denotes winding number.

II. BACKGROUND

Let M_1 and M_2 be complete, separable metric (Polish) spaces equipped with p^{th} order distance metric, say the L^p norm. Then the Wasserstein distance of order q, denoted as $_pW_q$, between two probability measures μ_1 and μ_2,

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supported on M_1 and M_2, is defined as

$$_pW_q(\mu_1,\mu_2) := \left[\inf_{\mu \in \mathcal{M}(\mu_1,\mu_2)} \int_{M_1 \times M_2} \|x-y\|_p^q \, d\mu(x,y) \right]^{1/q},$$

where $\mathcal{M}(\mu_1,\mu_2)$ is the set of all probability measures on $M_1 \times M_2$ with first marginal μ_1 and second marginal μ_2. It's well known [8] that on the set of Borel measures on $\mathbb{R}^d$ having finite second moments, $_pW_q$ defines a metric. If the measures μ_1 and μ_2 are absolutely continuous w.r.t. the Lebesgue measure, with densities ρ_1 and ρ_2, then we can write $\mathcal{M}(\rho_1,\rho_2)$ for the set $\mathcal{M}(\mu_1,\mu_2)$, and accordingly $_pW_q(\rho_1,\rho_2)$ in lieu of $_pW_q(\mu_1,\mu_2)$. This is assumed to hold for all subsequent analysis.

The Wasserstein metric is an integral notion of distance, as opposed to a pointwise notion (e.g. Hellinger distance, Kullback-Leibler (KL) divergence etc.), on the manifold of probability densities. This makes Wasserstein distance a good candidate for model validation, since the supports of the PDFs under consideration, evolving under two different dynamics, are often not identical. However, computation of Wasserstein metric is not straightforward. For real line, a closed form solution exists [9] in terms of the cumulative distribution functions (CDFs) of the test PDFs. Let F and G be the corresponding CDFs of the univariate PDFs ρ_1 and ρ_2 respectively. Then

$$_pW_q^q(\rho_1,\rho_2) = \int_0^1 \|F^{-1}(\varsigma) - G^{-1}(\varsigma)\|_p^q \, d\varsigma. \tag{1}$$

For multivariate case, in general, one has to compute $_pW_q$ from its definition, which can be cast as a linear program (LP) in mn variables with $(m+n+mn)$ constraints, where the respective PDFs have m and n-sample representations. This was detailed in [1], with $p = q = 2$. The choice $p = 2$ is due to the fact that we measure inter-sample distance in Euclidean metric. The choice $q = 2$ ensures good regularity properties for the metric.

III. FREQUENCY DOMAIN FORMULAE FOR W

A. SISO Case

Theorem 1: Consider two stable LTI systems with transfer functions G_i, $i = 1, 2$, both excited by stationary Gaussian input $u(t) \sim \mathcal{N}(\mu_u, \sigma_u^2)$, with power spectral density (PSD) $S_u(\omega)$. Then the Wasserstein gap W between them, is given by

$$W(G_1,G_2) = \sqrt{\mu_u^2(G_1(0) - G_2(0))^2 + (\sigma_1 - \sigma_2)^2}, \tag{2}$$

with $\sigma_i = \sqrt{\int_{-\infty}^{+\infty} |G_i(j\omega)|^2 S_u(\omega)\, d\omega - \mu_u^2 G_i^2(0)}$, $i = 1, 2$.

Proof: Let $y_i(t)$, $i = 1, 2$, denote the sample paths of the respective stationary outputs, with densities $\mathcal{N}(\mu_i, \sigma_i^2)$. If $R_{y_i}(\tau)$ are the respective autocorrelations with τ being the corresponding correlation window, then stationarity implies

$$R_{y_i}(\tau) = \mathbb{E}[y_i(t+\tau)y_i(t)] \Rightarrow R_{y_i}(0) = \mathbb{E}\left[(y_i(t))^2\right] \Rightarrow$$

$$\int_{-\infty}^{+\infty} S_{y_i}(\omega)\, d\omega = \int_{-\infty}^{+\infty} |G_i(j\omega)|^2 S_u(\omega)\, d\omega = \mu_i^2 + \sigma_i^2, \tag{3}$$

where $S_{y_i}(\omega)$ are the auto-PSDs, defined as the inverse Fourier transform of the respective auto-correlations, and we have used the well-known [10] SISO PSD relation $S_{y_i}(\omega) = |G_i(j\omega)|^2 S_u(\omega)$. On the other hand,

$$\mu_i = \mathbb{E}[y_i(t)] = \mathbb{E}\left[\int_{\infty}^{+\infty} h_i(\tau)u(t-\tau)\, d\tau\right],$$

where $h_i(.)$ are the respective impulse response functions. By changing the order of integrals, and using the fact that $\mathbb{E}[u(t-\tau)] = \mathbb{E}[u(t)]$ (due to stationarity), we get

$$\mu_i = \mu_u \int_{-\infty}^{+\infty} h_i(\tau) e^{-j.0.\tau} d\tau = \mu_u G_i(0). \tag{4}$$

Notice that (3) and (4) yield four equations in four unknowns μ_i, σ_i, $i = 1, 2$. Since Wasserstein distance between two univariate Gaussians $\mathcal{N}(\mu_i, \sigma_i^2)$ is given by [11]

$$W = \sqrt{(\mu_1 - \mu_2)^2 + (\sigma_1 - \sigma_2)^2}, \tag{5}$$

the result follows from (3) and (4). The input PSD $S_u(\omega)$ is a function of σ_u, depending on the temporal properties of the stochastic process $u(t)$. ∎

Corollary 2: If $u(t)$ is Gaussian *white* noise, then $\mu_u = 0$, $S_u(\omega) = \sigma_u^2 = $ constant. Consequently, W *is the difference between the respective $\mathcal{H}_2$ norms*, up to scaling by the strength of input disturbance, i.e.

$$W = \sqrt{2\pi}\sigma_u \left| \|G_1(j\omega)\|_2 - \|G_2(j\omega)\|_2 \right|. \tag{6}$$

The proof follows immediately from the definition of SISO $\mathcal{H}_2$ norm: $\|G_i\|_2 := \sqrt{\dfrac{1}{2\pi} \int_{-\infty}^{+\infty} G_i^*(j\omega) G_i(j\omega)\, d\omega}$, $i = 1, 2$. Notice that, in the definitions of Fourier transform pairs auto-correlation and PSD, the factor $\frac{1}{\sqrt{2\pi}}$ is usually omitted in the signal processing community [10], [12], and we have adopted the same convention in proving Theorem 1. Thus, by scaling the variance of the input noise, one can normalize the factor $\sqrt{2\pi}\sigma_u$ in (6), a condition we will assume in most derivations without loss of generality.

In the remaining of this paper, we will derive the results assuming the input to be Gaussian white noise. Given the input auto and cross-PSDs, how to handle more general cases, will become apparent from the proofs.

B. MISO Case

Theorem 3: Consider two stable LTI systems with m inputs and single output, having transfer arrays G and $\widehat{G}$, each being a row vector of size $1 \times m$. If both the systems are excited by Gaussian white noise vector $u(t) \sim \mathcal{N}(0_{m \times 1}, \text{diag}(\sigma_u^2))$, then the Wasserstein gap W between them, is given by the scaled difference between respective $\mathcal{H}_2$ norms:

$$W\left(G, \widehat{G}\right) = \sqrt{2\pi}\sigma_u \left| \|G(j\omega)\|_2 - \|\widehat{G}(j\omega)\|_2 \right|. \tag{7}$$

Proof: Like the SISO proof, we still have $\int_{-\infty}^{+\infty} S_y(\omega)\, d\omega = \sigma^2$, since $\mu = \widehat{\mu} = 0$, due to whiteness of the input. This eqn. holds true for both pairs $\left(S_y(\omega), \sigma^2\right)$

and $\left(S_{\widehat{y}}\left(\omega\right),\widehat{\sigma}^2\right)$. For the general correlated stationary input, the MISO PSD relation is known [10] to be

$$S_y\left(\omega\right) = \sum_{i=1}^{m}\sum_{k=1}^{m} G_i^*\left(j\omega\right) G_k\left(j\omega\right) \underbrace{S_{u_i u_k}\left(\omega\right)}_{\text{input PSD matrix}} d\omega. \quad (8)$$

Now, for white input vector, each dimension is an independent white noise process, implying the dimensions are mutually uncorrelated. Hence, $R_{u_i u_k}\left(\tau\right), S_{u_i u_k}\left(\omega\right) = 0, \forall i \neq k$. Thus, for Gaussian white vector $u\left(t\right) \sim \mathcal{N}\left(0_{m\times 1}, \text{diag}\left(\sigma_i^2\right)\right)$, (5) results

$$W = \left| \sqrt{\int_{-\infty}^{+\infty}\sum_{i=1}^{m}|G_i\left(j\omega\right)|^2\sigma_{u_i}^2 d\omega} - \sqrt{\int_{-\infty}^{+\infty}\sum_{i=1}^{m}|\widehat{G}_i\left(j\omega\right)|^2\sigma_{u_i}^2 d\omega} \right|$$

which reduces to (7) for *spherical* Gaussian case, since the $\mathcal{H}_2$ norm for multivariate case, is defined as $||G||_2 :=$ $\sqrt{\dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{+\infty} \text{tr}\left(G^H\left(j\omega\right) G\left(j\omega\right)\right) d\omega}$. ∎

C. MIMO Case

Theorem 4: Consider two stable LTI systems with m inputs and p outputs, having transfer matrices G and $\widehat{G}$. If both the systems are excited by Gaussian white noise vector $u\left(t\right) \sim \mathcal{N}\left(0_{m\times 1}, \text{diag}\left(\sigma_u^2\right)\right)$, then the Wasserstein gap W between them, is given by

$$W\left(G,\widehat{G}\right) = \sqrt{2\pi}\sigma_u \left(||G\left(j\omega\right)||_2^2 + ||\widehat{G}\left(j\omega\right)||_2^2 - 2\,\text{tr}\left[\left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} G^H\left(j\omega\right) G\left(j\omega\right) d\omega\right)^{1/2} \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} \widehat{G}^H\left(j\omega\right) \widehat{G}\left(j\omega\right) d\omega\right) \right.\right.$$
$$\left.\left. \left(\frac{1}{2\pi}\int_{-\infty}^{+\infty} G^H\left(j\omega\right) G\left(j\omega\right) d\omega\right)^{1/2} \right]^{1/2} \right)^{1/2}. \quad (9)$$

Proof: In this case, the output correlation matrices of size $p \times p$ satisfy $\mathbf{R}_y\left(0\right) = \int_{-\infty}^{+\infty} \mathbf{S}_y\left(\omega\right) d\omega = \Sigma_y$. Similar equation holds for the "hat" system.

Let the $(i,k)^{\text{th}}$ elements $G_{ik}, \widehat{G}_{ik}$, of the transfer matrices relate the response from i^{th} input to the respective k^{th} output. Then the MIMO PSD relation [13] becomes

$$\mathbf{S}_y\left(\omega\right) = G^H\left(j\omega\right) \underbrace{\mathbf{S}_u\left(\omega\right)}_{\text{Input PSD matrix}} G\left(j\omega\right), \quad (10)$$

where for Gaussian white input, we have $\mathbf{S}_u\left(\omega\right) = \sigma_u^2 I_{m\times m} \Rightarrow \mathbf{S}_y\left(\omega\right) = \sigma_u^2 G^H\left(j\omega\right) G\left(j\omega\right)$. Thus, $\Sigma_y = \sigma_u^2 \int_{-\infty}^{+\infty} G^H\left(j\omega\right) G\left(j\omega\right) d\omega$. Hence, $\text{tr}\left(\Sigma_y\right) = 2\pi\sigma_u^2||G||_2^2$; $\text{tr}\left(\Sigma_{\widehat{y}}\right) = 2\pi\sigma_u^2||\widehat{G}||_2^2$. Since $\mu_y = \mu_{\widehat{y}} = 0$, the Wasserstein distance between $y\left(t\right) \sim \mathcal{N}\left(\mu_y, \Sigma_y\right)$ and $\widehat{y}\left(t\right) \sim \mathcal{N}\left(\mu_{\widehat{y}}, \Sigma_{\widehat{y}}\right)$ is given by [11]

$$W = \sqrt{\text{tr}\left(\Sigma_y\right) + \text{tr}\left(\Sigma_{\widehat{y}}\right) - 2\,\text{tr}\left[\sqrt{\Sigma_y}\Sigma_{\widehat{y}}\sqrt{\Sigma_y}\right]^{1/2}}. \quad (11)$$

Substituting for Σ_y and $\Sigma_{\widehat{y}}$ in (11), in terms of G and $\widehat{G}$, we arrive at (9). ∎

D. Bounds for MIMO Case

Following results provide simpler and easier-to-interpret bounds for W in the MIMO case.

1) Lower bound:

Lemma 1: Given stable transfer matrices G and $\widehat{G}$, the Wasserstein distance for MIMO case, is lower bounded by the corresponding expression for SISO or MISO case, i.e. $\left|||G||_2 - ||\widehat{G}||_2\right| \leqslant W$.

Proof: Since Σ_y and $\Sigma_{\widehat{y}}$ are positive semi-definite, they satisfy (p. 527, Fact 8.12.20, [14])

$$\text{tr}\left(\sqrt{\Sigma_y}\Sigma_{\widehat{y}}\sqrt{\Sigma_y}\right)^{1/2} \leqslant \sqrt{\text{tr}\left(\Sigma_y\right)\text{tr}\left(\Sigma_{\widehat{y}}\right)}.$$

Since $\text{tr}\left(\Sigma_y\right) = ||G||_2^2$ and $\text{tr}\left(\Sigma_{\widehat{y}}\right) = ||\widehat{G}||_2^2$, we get

$$\underbrace{\text{tr}\left(\Sigma_y\right) + \text{tr}\left(\Sigma_{\widehat{y}}\right) - 2\,\text{tr}\left(\sqrt{\Sigma_y}\Sigma_{\widehat{y}}\sqrt{\Sigma_y}\right)^{1/2}}_{W^2} \geqslant \left(||G||_2 - ||\widehat{G}||_2\right)^2.$$

Hence the result. ∎

2) Upper bound:

Lemma 2: If Σ_y and $\Sigma_{\widehat{y}}$ are the output covariance matrices corresponding to stable transfer matrices G and $\widehat{G}$ respectively, then we have the following upper bound for MIMO Wasserstein distance: $W \leqslant \left|\left|\sqrt{\Sigma_y} - \sqrt{\Sigma_{\widehat{y}}}\right|\right|_F$.

Proof: It is known (Fact 8.19.21, [14]) that for $0 \leqslant p \leqslant 1$, $\text{tr}\left(\Sigma_y^p\Sigma_{\widehat{y}}^p\right) \leqslant \text{tr}\left(\Sigma_{\widehat{y}}^{1/2}\Sigma_y\Sigma_{\widehat{y}}^{1/2}\right)^p$. Taking $p = \frac{1}{2}$, we get

$$\text{tr}\left(\sqrt{\Sigma_y}\sqrt{\Sigma_{\widehat{y}}}\right) \leqslant \left(\sqrt{\Sigma_{\widehat{y}}}\Sigma_y\sqrt{\Sigma_{\widehat{y}}}\right)^{1/2} = \left(\sqrt{\Sigma_y}\Sigma_{\widehat{y}}\sqrt{\Sigma_y}\right)^{1/2},$$

where the last equality follows from the symmetry of Wasserstein distance, and can be separately proved by noting that $\text{tr}\left(\sqrt{MM^T}\right) = \text{tr}\left(\sqrt{M^TM}\right)$ for $M = \sqrt{\Sigma_y}\sqrt{\Sigma_{\widehat{y}}}$. Since $\text{tr}\left(\Sigma_y\right) = ||G||_2^2$ and $\text{tr}\left(\Sigma_{\widehat{y}}\right) = ||\widehat{G}||_2^2$, we obtain

$$W \leqslant \underbrace{\left(\text{tr}\left(\Sigma_y\right) + \text{tr}\left(\Sigma_{\widehat{y}}\right) - 2\,\text{tr}\left(\sqrt{\Sigma_y}\sqrt{\Sigma_{\widehat{y}}}\right)\right)^{1/2}}_{||\sqrt{\Sigma_y} - \sqrt{\Sigma_{\widehat{y}}}||_F}.$$

∎

IV. SENSITIVITY OF W IN FREQUENCY DOMAIN

One may interpret the frequency domain formulae for W derived above, as the difference between average gains of the two systems. Following are some observations regarding the same.

A. Scaling

W is sensitive to scaling. For example, if we set $\widehat{G} = kG$, where k is some non-zero scaling constant, then (6), (7) and (9) yields $W = |1 - k| \, ||G||_2$ (assuming $\sigma_u = 1/\sqrt{2\pi}$). Thus a linear relative amplification between two stable LTI systems, results a linear amplification of the Wasserstein gap. This can be contrasted with some recent works [15], [16] in the literature, on defining gap between dynamical

systems, where a gap was shown to be either insensitive [15] to scaling, or a non-linear function [16] of the scaling constant. As pointed out in [17], which one is a desirable property depends on the application.

B. Minimum versus Non-minimum Phase Systems

The frequency domain expressions for W depend only on the *magnitudes* of transfer functions. Thus, if we compare stable transfer functions of the form $G_{\pm} = \dfrac{\prod_{i=1}^{n_z}(s \pm z_i)}{\prod_{k=1}^{n_p}(s + p_k)}$ with $n_p > n_z$, and $\mathrm{Re}\,(p_k) > 0\ \forall k$, then $W\,(G_+, G_-) = 0$. This result is intuitively consistent (see the discussion at p. 1592 in [16]). However, the non-minimum phase zeros are related to $\mathcal{H}_2$ norms of the respective transfer functions via *Poisson-Jensen half-plane formula* (Appendix C.8.2 in [19]).

C. SISO Invariance Properties

Being a function of magnitudes only, the Wasserstein distance, like the chordal metric [18], remains invariant under complex conjugate transformation, i.e. $W\left(G, \widehat{G}\right) = W\left(G^*, \widehat{G}^*\right)$. However, unlike chordal metric, $W\left(G, \widehat{G}\right) \neq W\left(\frac{1}{G}, \frac{1}{\widehat{G}}\right)$, in general.

V. Geometric Interpretation of SISO Formula and Comparison with ν-gap Metric

The ν-gap metric [20]–[22], was introduced as an important tool for linear model validation with good robustness properties, and its extensions have been proposed [23], [24] for nonlinear systems. It's natural to ask how the Wasserstein distance, proposed in time domain [1] for both linear and nonlinear systems, relate with ν-gap. Like the SISO ν-gap, we look for a geometric interpretation of the SISO formula (6), which may be helpful for comparison between the metrics.

A. SISO ν-gap and W

Given two transfer functions G_1 and G_2, let $G_i = N_i M_i^{-1} = \widetilde{M}_i^{-1}\widetilde{N}_i$, $i = 1, 2$, denote the normalized right and left coprime factorizations [25], respectively. Further, let $\Gamma_i(s) := \{N_i(s)\, M_i(s)\}^T$, and $\widetilde{\Gamma}_i(s) := \{\widetilde{M}_i(s) - \widetilde{N}_i(s)\}$. If $\mathrm{wno}(\det(\Gamma_2^*(j\omega)\,\Gamma(j\omega))) \neq 0$, then the SISO ν-gap metric δ_ν is given by

$$\delta_\nu = \sup_{\omega \in \mathbb{R} \cup \{\infty\}} \frac{|G_1(j\omega) - G_2(j\omega)|}{\sqrt{1 + |G_1(j\omega)|^2}\sqrt{1 + |G_2(j\omega)|^2}}, \quad (12)$$

and lies between 0 to 1. When the winding number condition is not satisfied, then $\delta_\nu := 1$. Geometrically, ν-gap measures the largest chordal distance $\kappa(\omega)$ between the Nyquist plots of G_1 and G_2, projected on the Riemann sphere (Fig. 2). On the other hand, (6) can be geometrically interpreted as the difference between the lengths of the r.m.s. distances to the Nyquist plots of G_1 and G_2, measured from the origin (Fig. 2).

One difficulty in directly comparing W and δ_ν is that (12) is normalized, but (6) is not. Hence we can either compare (6) with the "un-normalized equivalent" of (12),

or we can normalize (6) and then compare with (12). In the latter case, as of yet, it's not clear what should be the intrinsic normalization factor. However, the geometric insight will guide us to answer both.

B. Comparison on the Complex Plane

Let $\kappa^{\mathrm{proj}}(\omega)$ be the projection of $\kappa(\omega)$ to the extended complex plane. For given transfer functions G and $\widehat{G}$, $\sup_\omega \kappa^{\mathrm{proj}}(\omega)$ is the largest pointwise distance between the two Nyquist plots, and can be taken as an "un-normalized analogue" of the largest normalized chord length δ_ν.

Theorem 5: Given two stable LTI transfer functions G and $\widehat{G}$, the difference between their r.m.s. lengths, is upper bounded by the maximum projected chordal length, i.e. $\sup_\omega \kappa^{\mathrm{proj}}(\omega) \geqslant W$.

Proof: The stereographic projection of a point $(x, y) := x + jy$ on the plane, to the point (ξ, η, ζ) on Riemann sphere, is given by [26]

$$\xi = \frac{x}{1 + x^2 + y^2}, \quad \eta = \frac{y}{1 + x^2 + y^2}, \quad \zeta = \frac{x^2 + y^2}{1 + x^2 + y^2}.$$

Form Fig. 2, $NG = \sqrt{1 + |G|^2}$, $N\widehat{G} = \sqrt{1 + |\widehat{G}|^2}$. Since $\dfrac{N\varphi(G)}{NG} = \dfrac{1 - \zeta}{1} = \dfrac{1}{1 + |G|^2}$, we have $N\varphi(G) = \left(1 + |G|^2\right)^{-1/2}$, and $N\varphi(\widehat{G}) = \left(1 + |\widehat{G}|^2\right)^{-1/2}$. Further, $N\varphi(G)\, NG = N\varphi(\widehat{G})\, N\widehat{G} = 1$ implies that triangles $N\varphi(G)\varphi(\widehat{G})$ and $NG\widehat{G}$ are similar. Consequently, $\dfrac{\kappa(\omega)}{\kappa^{\mathrm{proj}}(\omega)} = \dfrac{N\varphi(\widehat{G})}{NG} = \dfrac{1}{(1 + |G|^2)^{1/2}\left(1 + |\widehat{G}|^2\right)^{1/2}}$. Thus, $\kappa^{\mathrm{proj}}(\omega) = |G - \widehat{G}|$. Now, notice that

$$\sup_\omega \kappa^{\mathrm{proj}}(\omega) = ||G - \widehat{G}||_\infty \geqslant ||G - \widehat{G}||_2 \geqslant |\,||G||_2 - ||\widehat{G}||_2\,|,$$

where the last step is the reverse triangle inequality. This completes the proof. ∎

C. Comparison on the Riemann Sphere

To make a stereographic projection of Wasserstein distance onto the Riemann sphere, we first consider projections of the individual r.m.s. lengths given by the $\mathcal{H}_2$ norms of G and $\widehat{G}$. The following lemma is relevant in this regard.

Lemma 3: (p. 40, Theorem 2.5.1, [27]) Under stereographic projection, circlines in the complex plane get projected to circles on the Riemann sphere and vice versa. For straight lines on the plane, the corresponding circles on Riemann sphere pass through the north pole.

Corollary 6: The stereographic projection of $\mathcal{H}_2$ norm of any LTI transfer function can be of length at most $\frac{\pi}{2}$.

Proof: From Lemma 3, all straight lines on the complex plane, passing through origin, must go through both north and south poles, i.e. will be meridians on Riemann sphere. Points on such straight lines, situated infinite extent away from the origin, under stereographic projection, approach the north pole from both sides of the Riemann sphere. Thus, a ray on the complex plane, with fixed end at the origin,
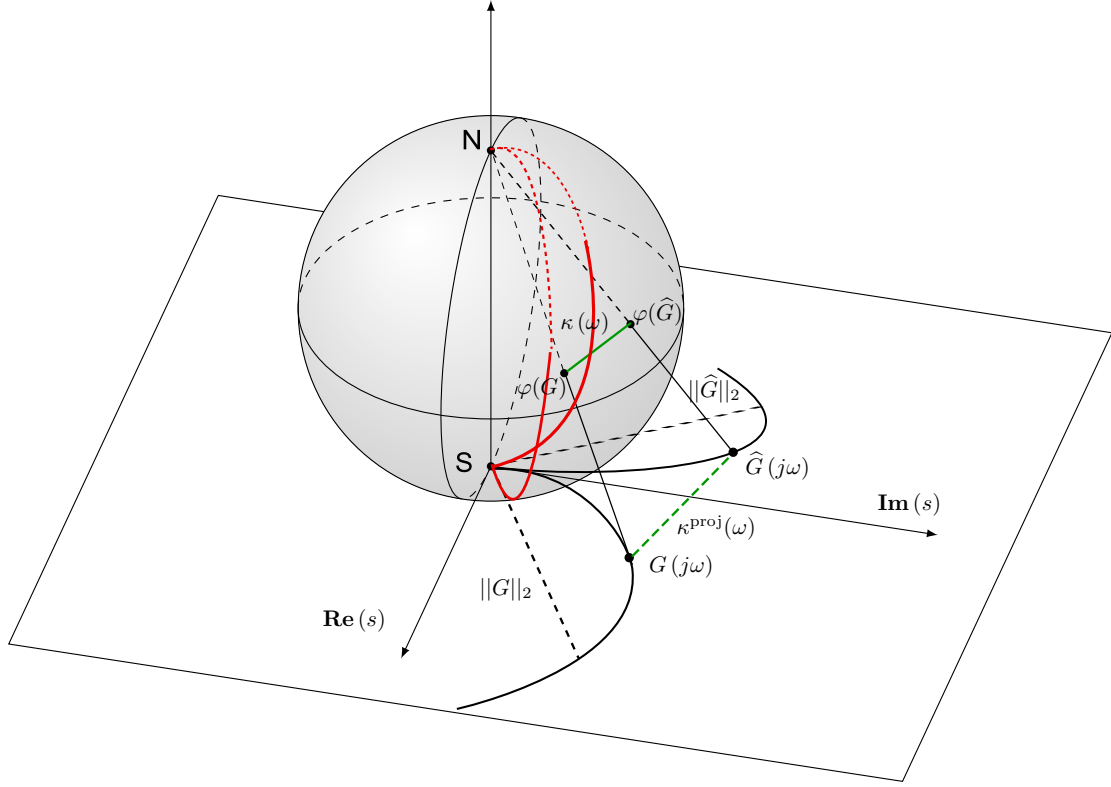
Fig. 1. The stereographic projection $\varphi : \mathbb{C} \cup \{\infty\} \mapsto \mathcal{S}$, where $\mathcal{S}$ is the Riemann sphere of unit diameter, with south pole at the origin of the extended complex plane. $\kappa^{\text{proj}}(\omega)$ (*dotted green*) is the projection of the chordal distance $\kappa(\omega)$ (*solid green*) to the plane. The *dotted black* lines on the plane denote the $\mathcal{H}_2$ norms, which measure the r.m.s. distances of the respective Nyquist plots (*solid black*) from the origin. The *solid red* arcs show the projections of the two $\mathcal{H}_2$ norms on the Riemann sphere. The projected W is the difference between these arc-lengths. The *dotted red* arcs show that values of the respective $\mathcal{H}_2$ norm projections can at most be $\frac{\pi}{2}$, resulting a normalization for projected W.

projects to half meridian of circumference at most $\frac{\pi}{2}$. This completes the proof. Notice that, half-meridian arc length on the Riemann sphere will be exactly $\frac{\pi}{2}$ iff $\mathcal{H}_2$ norm is infinity, *either* due to unstable transfer function *or* due to non-zero feed-through. ∎

Theorem 7: Given two LTI transfer functions G and $\widehat{G}$, the normalized Wasserstein distance $\overline{W}_{\mathcal{S}}$ on the Riemann sphere $\mathcal{S}$, is given by

$$\overline{W}_{\mathcal{S}}\left(G, \widehat{G}\right) = \frac{2}{\pi}\left| \arctan\|G\|_2 - \arctan\|\widehat{G}\|_2 \right|. \quad (13)$$

Proof: If the $\mathcal{H}_2$ norm of the transfer function is finite, the stereographic projection maps a line segment on complex plane (Fig. 2) to an arc of the half-meridian. The infinitesimal lengths on the plane and on the sphere relate [27] by

$$\frac{dl_{\mathcal{S}}}{dl_{\mathbb{C}\cup\infty}} = \frac{1}{1 + x^2 + y^2}. \quad (14)$$

Taking infinitesimal elements dr and $d\widehat{r}$ along the $\mathcal{H}_2$ norms (Fig. 2), (14) yields the half-meridial arc lengths

$$s = \int_0^{\|G\|_2} \frac{dr}{1 + r^2} = \arctan\|G\|_2,$$

$$\widehat{s} = \int_0^{\|\widehat{G}\|_2} \frac{d\widehat{r}}{1 + \widehat{r}^2} = \arctan\|\widehat{G}\|_2.$$

Since $0 \leqslant \|G\|_2, \|\widehat{G}\|_2 \leqslant \infty \Rightarrow 0 \leqslant s, \widehat{s} \leqslant \frac{\pi}{2}$, therefore $|s - \widehat{s}| \leqslant \frac{\pi}{2}$. Assuming the scaling $\sqrt{2\pi}\sigma_u = 1$ in (6), we

get the projected Wasserstein distance $W_{\mathcal{S}} = |s - \widehat{s}| = \left| \arctan\|G\|_2 - \arctan\|\widehat{G}\|_2 \right|$, which can be normalized by $\frac{\pi}{2}$ to result $\overline{W}_{\mathcal{S}}$. Notice that, either of the $\mathcal{H}_2$ norms can be infinity. ∎

The following theorem provides an indirect comparison between $\overline{W}_{\mathcal{S}}$ and δ_ν. It presents only sufficiency condition.

Theorem 8: Given stable LTI transfer functions G and $\widehat{G}$, let P and $\widehat{P}$ be the points on the respective Nyquist plots corresponding to their $\mathcal{H}_2$ norms. Let $\gamma_2 := \angle NPS$, $\widehat{\gamma}_2 := \angle N\widehat{P}S$. Similarly, define angles $\gamma(\omega)$ and $\widehat{\gamma}(\omega)$, for generic points $G(j\omega)$ and $\widehat{G}(j\omega)$. If $\left| \|\cos\gamma\sin\widehat{\gamma}\|_2 - \|\sin\gamma\cos\widehat{\gamma}\|_2 \right| \geqslant \frac{2}{\pi}|\gamma_2 - \widehat{\gamma}_2|$, then $\delta_\nu \geqslant \overline{W}_{\mathcal{S}}$.

Proof: We observe that

$$\delta_\nu = \sup_\omega \frac{|G - \widehat{G}|}{\sqrt{1 + |G|^2}\sqrt{1 + |\widehat{G}|^2}} = \left\| \frac{|G - \widehat{G}|}{\sqrt{1 + |G|^2}\sqrt{1 + |\widehat{G}|^2}} \right\|_\infty,$$

$$\geqslant \left\| \frac{|G - \widehat{G}|}{\sqrt{1 + |G|^2}\sqrt{1 + |\widehat{G}|^2}} \right\|_2,$$

$$\geqslant \left| \left\| \frac{|G|}{\sqrt{1 + |G|^2}\sqrt{1 + |\widehat{G}|^2}} \right\|_2 - \left\| \frac{|\widehat{G}|}{\sqrt{1 + |G|^2}\sqrt{1 + |\widehat{G}|^2}} \right\|_2 \right|, \quad (15)$$

where the last step follows form triangle difference inequality for $\mathcal{H}_2$ norms.

Notice that, γ_2 and $\widehat{\gamma}_2$ are the angles subtended by the respective $\mathcal{H}_2$ norms with its stereographic projections. Clearly, $0 \leqslant \gamma_2, \widehat{\gamma}_2 \leqslant \frac{\pi}{2}$, and we can rewrite (13) as $\frac{2}{\pi}\left|\gamma_2 - \widehat{\gamma}_2\right|$. Likewise, we define the running angles γ and $\widehat{\gamma}$ as functions of ω, associated with points $G(j\omega)$ and $\widehat{G}(j\omega)$ (see Fig. 2). Further, notice that

$$\cos\gamma = \frac{|G|}{\sqrt{1+|G|^2}}, \quad \sin\widehat{\gamma} = \frac{1}{\sqrt{1+|\widehat{G}|^2}}, \quad (16)$$

$$\cos\widehat{\gamma} = \frac{|\widehat{G}|}{\sqrt{1+|\widehat{G}|^2}}, \quad \sin\gamma = \frac{1}{\sqrt{1+|G|^2}}, \quad (17)$$

and consequently, (15) can be written as the *difference between the r.m.s. values* of $\cos\gamma\sin\widehat{\gamma}$ and $\sin\gamma\cos\widehat{\gamma}$. Hence the result. $\blacksquare$

VI. CONCLUSIONS AND FUTURE WORK

In this paper, we have provided frequency domain formulae for Wasserstein distance, which has a natural time-domain interpretation originated from the theory of optimal transport. The geometric characterization of such formula was also derived and used to compare with ν-gap metric. It remains to see if one could derive a normalized coprime factor characterization of the MIMO W.

In contrast with ν-gap metric δ_ν, and gap metric δ_g (see [28]–[30]), W defined as is, measures the open-loop gap, since both systems are excited by the same input. A small value of $\delta_\nu\left(G,\widehat{G}\right)$ or $\delta_g\left(G,\widehat{G}\right)$ implies that any controller that works satisfactorily with one system, will do well for the other system too, even though the frequency responses of the open-loop plants may differ significantly. On the other hand, a small value of W implies closeness of their average gains. This dichotomy brings up an interesting direction of research, namely closed-loop validation in W. For example, given two open-loop systems are within ϵ levels of W, if the $\mathcal{H}_2$-optimal controller for one system is also wrapped with the other, then under what conditions the resulting closed loop systems will still be within ϵ-level of W. Instead of $\mathcal{H}_2$ optimality, other conditions of performance, e.g. set of all stabilizing controllers, $\mathcal{H}_\infty$ optimality could be investigated. It will also be of interest, if these ideas could lead to model reduction in W.

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